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Note

Matroids with few non-common bases☆

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Abstract

In [On Mills's conjecture on matroids with many common bases, Discrete Math. 240 (2001) 271–276], Lemos proved a conjecture of Mills [On matroids with many common bases, Discrete Math. 203 (1999) 195–205]: for two $(k+1)$-connected matroids whose symmetric difference between their collections of bases has size at most k, there is a matroid that is obtained from one of these matroids by relaxing n_1 circuit-hyperplanes and from the other by relaxing n_2 circuit-hyperplanes, where n_1 and n_2 are non-negative integers such that $n_1+n_2 \leqslant k$. In [Matroids with many common bases, Discrete Math. 270 (2003) 193–205], Lemos proved a similar result, where the hypothesis of the matroids being k-connected is replaced by the weaker hypothesis of being vertically k-connected. In this paper, we extend these results.

Keywords: Matroid; Bases; Connectivity

1. Introduction

After writing [1,2], we realized that the hypotheses of many theorems that appear in these papers can be significantly weakened to yield essentially the same conclusions. To prove the new results, the statement of some lemmas of [1,2] need to be changed but the same proofs given before work. Therefore we do not rewrite these proofs in this paper.

In this article, for matroid theory we shall use the notation set by Oxley in [6], which we assume familiar to the reader. We define the *connectivity function* of a matroid M as

$$\xi_M(X,Y) = r(X) + r(Y) - r(M),$$

for a partition $\{X, Y\}$ of $E(M)$. Tutte [9] said that a matroid M is *n-connected*, for a positive integer n, provided

$$\xi_M(X,Y) \geqslant k$$

for every integer k and partition $\{X, Y\}$ of $E(M)$ such that $k < n$ and

$$\min\{|X|, |Y|\} \geqslant k.$$

Lemos [1] proved, up to small modifications, the following conjecture made by Mills [3].

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Conjecture 1.1. Let k be a positive integer. Suppose M_1 and M_2 are $(k+1)$-connected matroids on a set E where $|E| \geqslant 2$. If $|\mathscr{B}(M_1) \triangle \mathscr{B}(M_2)| = k$, then there is an integer $j \in [0, k]$ and a matroid N on E that is obtained from M_1 and M_2 by relaxing j and $k-j$ circuit-hyperplanes, respectively.

Observe that, for $r \geqslant 1$, $U_{r-1,2r-1}$ and $U_{r,2r-1}$ are m-connected matroids for every positive integer m. As these matroids have different rank, it follows that

$$|\mathscr{B}(U_{r-1,2r-1}) \triangle \mathscr{B}(U_{r,2r-1})| = |\mathscr{B}(U_{r-1,2r-1}) \cup \mathscr{B}(U_{r,2r-1})| = 2\binom{2r-1}{r} = f(r).$$

When $k = f(r)$, this pair of matroids is a counter-example to Conjecture 1.1. Lemos [1] proved that these are the only exceptions and that the hypothesis on the size of E can be removed from this conjecture.

In [3], Mills proved this conjecture for $k=2$ and pointed out that it follows from Truemper's results when $k=1$ [7]. In general, this conjecture follows from a theorem which guarantees the same conclusion with weaker hypotheses. To state this result, we need some definitions.

For a matroid M, the *girth* of M is defined as

$$g(M) = \min\{|C| : C \in \mathscr{C}(M)\},$$

where this minimum is taken to be 0, when $\mathscr{C}(M) = \emptyset$. Lemos [1] proved:

Theorem 1.1. *For some positive integer k, suppose that M_1 and M_2 are matroids having the same ground set such that*

$$|\mathscr{B}(M_1) \triangle \mathscr{B}(M_2)| \leqslant k.$$

If $\min\{g(M_1), g(M_1^*)\} \geqslant k+1$, *then there is a matroid N which is obtained from M_1 and M_2 by relaxing n_1 and n_2 circuit-hyperplanes, respectively, where n_1 and n_2 are non-negative integers such that*

$$n_1 + n_2 = |\mathscr{B}(M_1) \triangle \mathscr{B}(M_2)| \leqslant k.$$

In [2], Lemos proved that this result is sharp. In this paper, we generalize this result by proving the next two theorems.

Theorem 1.2. *For some positive integer k, suppose that M_1 and M_2 are matroids having the same ground set such that*

$$n = |\mathscr{B}(M_1) - \mathscr{B}(M_2)| \leqslant k.$$

If $\min\{g(M_1), g(M_1^*)\} \geqslant k+1$, *then there is a matroid N which is obtained from M_2 by relaxing n circuit-hyperplanes such that $\mathscr{B}(M_1) \subseteq \mathscr{B}(N) = \mathscr{B}(M_1) \cup \mathscr{B}(M_2)$.*

Theorem 1.3. *For some positive integer k, suppose that M_1 and M_2 are matroids having the same ground set such that*

$$\max\{|\mathscr{B}(M_1) - \mathscr{B}(M_2)|, |\mathscr{B}(M_2) - \mathscr{B}(M_1)|\} \leqslant k.$$

If $\min\{g(M_1), g(M_1^*)\} \geqslant k+1$, *then there is a matroid N which is obtained from M_1 and M_2 by relaxing n_1 and n_2 circuit-hyperplanes, respectively, where n_1 and n_2 are non-negative integers such that*

$$n_1 = |\mathscr{B}(M_2) - \mathscr{B}(M_1)| \quad \textit{and} \quad n_2 = |\mathscr{B}(M_1) - \mathscr{B}(M_2)|.$$

Observe that Theorem 1.1 is a consequence of Theorem 1.3. Note that Theorem 1.3 is a consequence of Theorem 1.2 only when $\min\{g(M_2), g(M_2^*)\} \geqslant k+1$ also holds.

An n-connected matroid M such that $|E(M)| \geqslant 2n-2$ must satisfy

$$\min\{g(M), g(M^*)\} \geqslant n. \tag{1.1}$$

As a consequence of Theorem 1.1, Lemos [1] obtained the following theorem that proves Mills's Conjecture.

Theorem 1.4. *For some positive integer k, suppose that M_1 and M_2 are $(k+1)$-connected matroids having the same ground set E. If $|\mathscr{B}(M_1)\triangle\mathscr{B}(M_2)| = k$, then either*

(i) *there is an integer $j \in [0, k]$ and a matroid N on E that is obtained from M_1 and M_2 by relaxing j and $k-j$ circuit-hyperplanes, respectively; or*
(ii) *there is a positive integer r such that*

$$2\binom{2r-1}{r} = k$$

and $\{M_1, M_2\} = \{U_{r-1,2r-1}, U_{r,2r-1}\}$.

In this paper, we prove the next two generalizations of this theorem. We obtain the same conclusion with much weaker hypotheses.

Theorem 1.5. *For some positive integer k, suppose that M_1 and M_2 are $(k+1)$-connected matroids having the same ground set. If $n = |\mathscr{B}(M_1) - \mathscr{B}(M_2)| \leqslant k$, then either*

(i) *there is a matroid N that is obtained from M_2 by relaxing n circuit-hyperplanes such that $\mathscr{B}(M_1) \subseteq \mathscr{B}(N) = \mathscr{B}(M_1) \cup \mathscr{B}(M_2)$; or*
(ii) *there is a positive integer r such that*

$$\binom{2r-1}{r} = n$$

and $\{M_1, M_2\} = \{U_{r-1,2r-1}, U_{r,2r-1}\}$.

Theorem 1.6. *For some positive integer k, suppose that M_1 and M_2 are $(k+1)$-connected matroids having the same ground set. If*

$$\max\{|\mathscr{B}(M_1) - \mathscr{B}(M_2)|, |\mathscr{B}(M_2) - \mathscr{B}(M_1)|\} \leqslant k,$$

then either

(i) *there is a matroid N on E that is obtained from M_1 and M_2 by relaxing $|\mathscr{B}(M_2) - \mathscr{B}(M_1)|$ and $|\mathscr{B}(M_1) - \mathscr{B}(M_2)|$ circuit-hyperplanes, respectively; or*
(ii) *there is a positive integer r such that*

$$\binom{2r-1}{r} = |\mathscr{B}(M_1) - \mathscr{B}(M_2)| = |\mathscr{B}(M_2) - \mathscr{B}(M_1)| \leqslant k$$

and $\{M_1, M_2\} = \{U_{r-1,2r-1}, U_{r,2r-1}\}$.

Observe that both Theorems 1.4 and 1.6 are consequences of Theorem 1.5.

Condition (1.1) is very strong. This happens because Tutte's definition of matroid n-connectedness has the attractive property of being invariant under duality but it does not generalize the notion of an n-connected graph. For such a generalization, we need a new definition (see [4,5]). We say that a matroid M is *vertically n-connected*, for a positive integer n, provided

$$\xi_M(X, Y) \geqslant k$$

for every integer k and partition $\{X, Y\}$ of $E(M)$ such that $k < n$ and

$$\min\{r(X), r(Y)\} \geqslant k.$$

Observe that every vertically n-connected matroid is n-connected, but the converse does not hold. When M is a vertically n-connected matroid such that $r(M) \geqslant n$, then

$$g(M^*) \geqslant n. \tag{1.2}$$

Note that this condition is much weaker than (1.1).

The next result was proved in [2]. To state it, we need to describe an operation that generalizes the relaxation of a circuit-hyperplane. We say that a hyperplane H of a matroid M is a *tip-hyperplane* having *tip* e provided $e \in H$, $r(M) \leqslant |H|$ and $M|H$ is obtained from $M|(H-e)$ by adding e freely. Hence the circuits of $M|H$ containing e are

$$\mathscr{C}_e(M|H) = \{B \cup \{e\} : B \in \mathscr{B}(M|(H-e))\}.$$

In particular, every circuit of $M|H$ that contains e has cardinality equal to $r(M)$. In [2], it was proved that $\mathscr{B}(M) \cup \mathscr{C}_e(M|H)$ is the set of bases of a matroid which we say is *obtained from M by relaxing the tip-hyperplane H*.

Theorem 1.7. *For some positive integer k, suppose that M_1 and M_2 are vertically $(k+1)$-connected matroids having the same ground set E. If $|\mathscr{B}(M_1) \triangle \mathscr{B}(M_2)| \leqslant k$, then*:

(i) *there is a matroid N on E that is obtained from each of M_1 and M_2 by relaxing a sequence of tip-hyperplanes. Moreover*,

$$\mathscr{B}(M_1) \triangle \mathscr{B}(M_2) = [\mathscr{B}(N) - \mathscr{B}(M_1)] \cup [\mathscr{B}(N) - \mathscr{B}(M_2)]; \ \textit{or}$$

(ii) *for each $i \in \{1, 2\}$, $r(M_i) \leqslant k$ and M_i does not have disjoint cocircuits*.

Suppose that M_1 and M_2 are matroids over the same ground set. When M_i does not have disjoint cocircuits, for each $i \in \{1, 2\}$, then M_i is a vertically n-connected matroid for every n. So, the pairs of matroids described in (ii) appear because we do not ask in the definition of a vertically n-connected matroid that its rank is at least n—that is, we permit small matroids to be vertically n-connected.

Observe that Theorem 1.4 is a consequence of this theorem. In this paper, we generalize this result by proving the next theorem.

Theorem 1.8. *For some positive integer k, suppose that M_1 and M_2 are vertically $(k+1)$-connected matroids having the same ground set E. If*

$$\max\{|\mathscr{B}(M_1) - \mathscr{B}(M_2)|, |\mathscr{B}(M_2) - \mathscr{B}(M_1)|\} \leqslant k,$$

then:

(i) *there is a matroid N on E that is obtained from each of M_1 and M_2 by relaxing a sequence of tip-hyperplanes. Moreover*,

$$\mathscr{B}(M_1) \triangle \mathscr{B}(M_2) = [\mathscr{B}(N) - \mathscr{B}(M_1)] \cup [\mathscr{B}(N) - \mathscr{B}(M_2)]; \ \textit{or}$$

(ii) *for each $i \in \{1, 2\}$, $r(M_i) \leqslant k$ and M_i does not have disjoint cocircuits.*

2. Some basic lemmas

For a non-negative integer k, we say that (M_1, M_2) is a

(i) *k-pair* when M_1 and M_2 are matroids over the same ground set and

$$|\mathscr{B}(M_1) \triangle \mathscr{B}(M_2)| \leqslant k.$$

(ii) *strong k-pair* when M_1 and M_2 are matroids over the same ground set and

$$\max\{|\mathscr{B}(M_1) - \mathscr{B}(M_2)|, |\mathscr{B}(M_2) - \mathscr{B}(M_1)|\} \leqslant k.$$

(iii) *skew k-pair* when M_1 and M_2 are matroids over the same ground set and

$$|\mathscr{B}(M_1) - \mathscr{B}(M_2)| \leqslant k.$$

Observe that: if (M_1, M_2) is a k-pair, then (M_1, M_2) is a strong k-pair; and (M_1, M_2) is a strong k-pair if and only if both (M_1, M_2) and (M_2, M_1) are skew k-pairs.

For a matroid M, the *circumference* of M is defined as

$$\mathrm{c}(M) = \max\{|C| : C \in \mathscr{C}(M)\},$$

where this maximum is taken to be 0, when $\mathscr{C}(M) = \emptyset$.

Lemma 2.1. *Let k be a positive integer. If (M_1, M_2) is a skew k-pair such that* $\mathrm{c}(M_1^*) \geqslant k+1$, *then* $r(M_1) = r(M_2)$.

Proof. Let C^* be a cocircuit of M_1 such that $|C^*| = \mathrm{c}(M_1^*)$. There is an independent set I of M_1 such that $|I| = r(M_1) - 1$ and $I \cap C^* = \emptyset$, since $E(M_1) - C^*$ is a hyperplane of M_1. If

$$\mathscr{B} = \{I \cup f : f \in C^*\},$$

then $|\mathscr{B}| \geqslant k+1$. Hence $\mathscr{B} \cap \mathscr{B}(M_2)$ is non-empty. Thus $r(M_1) = r(M_2)$ and the result follows. □

The same proofs of Lemma 2 and Lemma 3 of [1] can be used to prove respectively Lemma 2.2 and Lemma 2.3. Consequently, we do not demonstrate the next two lemmas.

Lemma 2.2. *Let k be a positive integer. If (M_1, M_2) is a skew k-pair such that* $\mathrm{g}(M_1^*) \geqslant k+1$, *then*

$$\mathscr{I}(M_1) \cap \mathscr{C}(M_2) \subseteq \mathscr{B}(M_1).$$

For a matroid M, we denote by $\mathscr{CH}(M)$ the set of circuit-hyperplanes of M.

Lemma 2.3. *Let k be a positive integer. If (M_1, M_2) is a skew k-pair such that* $\min\{\mathrm{g}(M_1), \mathrm{g}(M_1^*)\} \geqslant k+1$, *then*

$$\mathscr{I}(M_1) \cap \mathscr{C}(M_2) = \mathscr{B}(M_1) \cap \mathscr{CH}(M_2).$$

Proof of Theorem 1.2. Let N be the matroid obtained from M_2 by relaxing all circuit-hyperplanes belonging to $\mathscr{B}(M_1) \cap \mathscr{CH}(M_2)$. By Lemma 2.3, $\mathscr{B}(M_1) \subseteq \mathscr{B}(N)$ and the result follows. □

Recall that a matroid M is *paving* if it has no circuits of size less than $r(M)$. The proof of Lemma 4 of [1] can be used to show the next result. Thus we omit its proof.

Lemma 2.4. *Let k be a positive integer. If (M_1, M_2) is a skew k-pair and*

$$\min\{\mathrm{g}(M_1), \mathrm{g}(M_1^*)\} \geqslant k+1,$$

then either

(i) $\min\{\mathrm{g}(M_2), \mathrm{g}(M_2^*)\} \geqslant k+1$, *or*
(ii) *M_1 is a uniform matroid and M_2 is a paving matroid having the same rank as M_1, which is equal to k or $|E(M_1)| - k$. Moreover,*

$$\{C : C \in \mathscr{C}(M_2) \text{ and } |C| = r(M_2)\} = \mathscr{CH}(M_2).$$

The same proof of Lemma 5 of [1] can be used to prove the next result and its proof is omitted:

Lemma 2.5. *Let k be a positive integer. If (M_1, M_2) is a strong k-pair and*

$$\min\{\mathrm{g}(M_1), \mathrm{g}(M_1^*)\} \geqslant k+1,$$

then

$$\mathscr{B}(M_i) - \mathscr{B}(M_{3-i}) = \mathscr{CH}(M_{3-i}) - \mathscr{CH}(M_i),$$

for every $i \in \{1, 2\}$.

Proof of Theorem 1.3. For $i \in \{1, 2\}$, let N_i be the matroid obtained from M_i by relaxing the n_i circuit-hyperplanes belonging to $\mathscr{CH}(M_i) - \mathscr{CH}(M_{3-i})$. By Lemma 2.5, $n_i = |\mathscr{B}(M_{3-i}) - \mathscr{B}(M_i)|$ and

$$\mathscr{B}(N_i) = \mathscr{B}(M_1) \cup \mathscr{B}(M_2).$$

Hence $N_1 = N_2$ and the result follows. $\square$

We say that a matroid M is *square*, when

$$|E(M)| - 2r(M) \in \{-1, 0, 1\}.$$

The next result follows from Corollary 8.1.8 of [6]:

Lemma 2.6. *If M is a $(k+1)$-connected matroid and $|E(M)| < 2k$, then M is a square uniform matroid.*

Proof of Theorem 1.5. When $|E(M)| \geqslant 2k$, then $\min\{g(M_1), g(M_1^*)\} \geqslant k+1$ and the result follows from Theorem 1.2. So we may suppose that $|E(M)| \leqslant 2k-1$. Then Lemma 2.6 implies that M_1 and M_2 are square uniform matroids. As $E(M_1) = E(M_2)$ and $M_1 \neq M_2$, it follows that $M_1 = M_2^*$ and $|E(M_1)| = 2r - 1$, for some positive integer r. Observe that

$$\mathscr{B}(M_1) - \mathscr{B}(M_2) = \mathscr{B}(M_1)$$

because $r(M_1) \neq r(M_2)$ and so $n = \binom{2r-1}{r}$. $\square$

3. Proving the other main result

For a matroid M, a subset L of $E(M)$ is said to be a *Tutte-line*, when $M|L$ has corank equal to two and no coloops. In [8], Tutte proved that L has a partition, which we call *the canonical partition of* L *in* M, $\{P_1, P_2, \ldots, P_n\}$, for some $n \geqslant 2$, such that $\mathscr{C}(M|L) = \{L - P_1, L - P_2, \ldots, L - P_n\}$. We denote by $\mathscr{TL}(M)$ the set of Tutte-lines of the matroid M. Following Tutte [8], we say that a set of circuits $\mathscr{L}$ of a matroid M is a *linear subclass* of M, when $\mathscr{C}(M|L) \subseteq \mathscr{L}$ for every $L \in \mathscr{TL}(M)$ such that $|\mathscr{C}(M|L) \cap \mathscr{L}| \geqslant 2$.

A circuit C of a matroid M is *large*, when $r(M) = |C|$. Let $\mathscr{LC}(M)$ be the set of large circuits of M. We say that Z is a *nest* of a matroid M, when $Z = \mathrm{cl}_M(C)$, for some $C \in \mathscr{LC}(M)$. Observe that $Z = \mathrm{cl}_M(C')$, for every $C' \in \mathscr{LC}(M)$ such that $C' \subseteq Z$. When a circuit-hyperplane C'' of M is contained in a nest Z of M, then $Z = C''$.

Suppose that C is a large circuit of a matroid M. Let Z be the nest of M that contains C. We consider a linear subclass $\mathscr{L}$ of $M|Z$ satisfying

$$\mathscr{L} \neq \mathscr{C}(M|Z) \text{ and } \mathscr{C}(M|Z) - \mathscr{HAM}(M|Z) \subseteq \mathscr{L},$$

where $\mathscr{HAM}(H)$ denotes the set of Hamiltonian circuits of a matroid H. A linear subclass of $M|Z$ satisfying these conditions will be called *admissible*. In Section 3 of [2], Lemos proved that

$$\mathscr{B}(M) \cup (\mathscr{C}(M|Z) - \mathscr{L})$$

is the set of bases of a matroid, which we shall denote by $M_{Z,\mathscr{L}}$. We say that this matroid is obtained from M by *relaxing the nest* Z *along the admissible linear subclass* $\mathscr{L}$ *of* $M|Z$. The next result is Lemma 3.3 of [2].

Lemma 3.1. *If Z' is a nest of M different of Z, then Z' is a nest of $M_{Z,\mathscr{L}}$. Moreover, $M|Z' = M_{Z,\mathscr{L}}|Z'$.*

This lemma is very important because we can fix a set of nests of M and for each nest in this family we can choose an admissible liner subclass of M restricted to this nest, when it exists. When we relax one of these nests along its admissible linear subclass, the other nests are nests of the resulting matroid and the admissible linear subclasses associated with them retains this property in the new matroid. So, we can continue with the process (and the order is irrelevant). Thus, this construction behaves similarly to the relaxing of a set of circuit-hyperplanes. Moreover, it agrees with it when every nest is a circuit-hyperplane.

We need the next lemma (Lemma 2.3 of [2]).

Lemma 3.2. *Let M_1 and M_2 be matroids having the same rank and ground set such that*

$$\mathscr{C} = \mathscr{B}(M_1) \triangle \mathscr{B}(M_2),$$

where $\mathscr{C}$ is the set of minimal elements belonging to $\mathscr{C}(M_1) \triangle \mathscr{C}(M_2)$. If $C \in \mathscr{C} \cap \mathscr{C}(M_1)$ and $Z = \mathrm{cl}_{M_1}(C)$, then

(i) *If L is a Tutte-line of M_1 such that $L \subseteq Z$ and $\mathscr{C} \cap \mathscr{C}(M_1|L) \neq \emptyset$, then L contains just one circuit C_L of M_2 and*

$$\mathscr{C}(M_1|L) - \{C_L\} = \{L - a : a \in C_L\} \subseteq \mathscr{C}.$$

Moreover, $C_L \in \mathscr{C}(M_1)$ provided $C_L \neq L$.

(ii) *$\mathscr{C}(M_1|Z) - \mathscr{C}$ is a linear subclass of $M_1|Z$.*

The next result shows that the hypotheses of the previous lemma are satisfied provided (M_1, M_2) is a strong k-pair such that $\min\{\mathrm{g}(M_1^*), \mathrm{g}(M_2^*)\} \geqslant k+1$.

Lemma 3.3. *Let k be a positive integer. If (M_1, M_2) is a strong k-pair such that $\min\{\mathrm{g}(M_1^*), \mathrm{g}(M_2^*)\} \geqslant k+1$, then*

$$\mathscr{B}(M_1) \triangle \mathscr{B}(M_2) = [\mathscr{I}(M_1) \cap \mathscr{C}(M_2)] \cup [\mathscr{C}(M_1) \cap \mathscr{I}(M_2)].$$

Proof. This result follows provided, for $i \in \{1, 2\}$,

$$\mathscr{I}(M_i) \cap \mathscr{C}(M_{3-i}) = \mathscr{B}(M_i) - \mathscr{B}(M_{3-i}). \tag{3.1}$$

By symmetry, we need to prove (3.1) only for $i = 1$. By Lemma 2.2, $\mathscr{I}(M_1) \cap \mathscr{C}(M_2) \subseteq \mathscr{B}(M_1)$ and so

$$\mathscr{I}(M_1) \cap \mathscr{C}(M_2) \subseteq \mathscr{B}(M_1) - \mathscr{B}(M_2). \tag{3.2}$$

If $B \in \mathscr{B}(M_1) - \mathscr{B}(M_2)$, then there is $C \in \mathscr{C}(M_2)$ such that $C \subseteq B$. Hence $C \in \mathscr{I}(M_1) \cap \mathscr{C}(M_2)$ and so $C \in \mathscr{B}(M_1)$, by Lemma 2.2. Thus $C = B$ and $B \in \mathscr{I}(M_1) \cap \mathscr{C}(M_2)$. Hence equality holds in (3.2). $\square$

The next result (Theorem 4.1 of [2]) plays a similar role in the proof of Theorem 1.7 as did Theorem 1.1 in the proof of Theorem 1.4.

Theorem 3.1. *Let k be a positive integer. If (M_1, M_2) is a k-pair and*

$$\min\{\mathrm{g}(M_1^*), \mathrm{g}(M_2^*)\} \geqslant k+1,$$

then there is a matroid N which is obtained from M_i, for both $i \in \{1, 2\}$, by relaxing a sequence of nests (each along an admissible linear subclass of M_i restricted to it) such that

$$\mathscr{B}(M_1) \triangle \mathscr{B}(M_2) = [\mathscr{B}(N) - \mathscr{B}(M_1)] \cup [\mathscr{B}(N) - \mathscr{B}(M_2)].$$

The proof of this theorem given in [2] can be used to show the next result:

Theorem 3.2. *Let k be a positive integer. If (M_1, M_2) is a strong k-pair and*

$$\min\{\mathrm{g}(M_1^*), \mathrm{g}(M_2^*)\} \geqslant k+1,$$

then there is a matroid N which is obtained from M_i, for both $i \in \{1, 2\}$, by relaxing a sequence of nests (each along an admissible linear subclass of M_i restricted to it) such that

$$\mathscr{B}(M_1)\triangle\mathscr{B}(M_2) = [\mathscr{B}(N) - \mathscr{B}(M_1)] \cup [\mathscr{B}(N) - \mathscr{B}(M_2)].$$

The proof of Theorem 1.8 is equal to the proof of Theorem 1.3 of [2] and it will be omitted.

We finish this paper proposing a conjecture:

Conjecture 3.1. For some positive integer k, suppose that M_1 and M_2 are vertically $(k+1)$-connected matroids having the same ground set E. If

$$|\mathscr{B}(M_1) - \mathscr{B}(M_2)| \leqslant k,$$

then:

(i) there is a matroid N on E that is obtained from M_2 by relaxing a sequence of tip-hyperplanes. Moreover,

$$\mathscr{B}(M_1) - \mathscr{B}(M_2) = \mathscr{B}(N) - \mathscr{B}(M_2); \text{ or}$$

(ii) for each $i \in \{1, 2\}$, $r(M_i) \leqslant k$ and M_i does not have disjoint cocircuits.

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